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FOCUSING EFFECTS IN TWO-DIMENSIONAL, SUPERSONIC FLOW

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The steady, supersonic, irrotational, isentropic, two-dimensional, shock-free flow of a perfect gas is investigated by a new, geometrical, method based on the use of characteristic co-ordinates. Some of the results apply also to more general problems of compressible flow involving two independent variables (§ 1).

The method is applied in particular to the treatment of the non-linear, non-analytic features. The variation in magnitude of discontinuities of the velocity gradient is determined as a function of the Mach number in § 4. The reflexion at the sonic line of such discontinuities is treated in § 7. The singularities of the field of flow are discussed in §§ 5 to 5·4; Craggs's (1948) results are extended to the case when the velocity components are not analytic functions of position, and to the case in which both the hodograph transformation and the inverse transformation are singular. Examples are given of singularities that occur in familiar flow problems, but have not hitherto been described (§§ 5·3, 5·4). Some properties are established of the geometry in the large of Mach line patterns; these properties are useful for the prediction of limit lines (§ 5·2).

The problem of the start of an oblique shockwave in the middle of the flow is briefly reviewed in § 6.

In the appendix it is shown that the conventional method of characteristics for the numerical treatment of two-dimensional, isentropic, irrotational, steady, supersonic flows must be modified near a branch line if a loss of accuracy is to be avoided.

1. Introduction

Several authors have employed characteristic co-ordinates, which are natural systems of co-ordinates for problems of supersonic flow. In the present investigation a consistent notation based on this idea is developed (§ 2). It provides a useful method for the qualitative discussion of supersonic flow fields, and a supplementary tool for quantitative investigations by the conventional method of characteristics.

The method consists, effectively, in the formulation of the flow problem as one of Riemannian geometry, with a metric tensor, which is unknown *a priori* and which has to be determined from the boundary conditions by the help of the focusing equations. A more elementary notation, however, has been adopted here. The physical concepts are translated into geometrical concepts at the beginning, and it is found that the argument is greatly simplified if it is carried on entirely in geometrical terms. Thus, the (non-linear) equations of motion are replaced by the (linear) focusing equations (§ 3) for the length parameters introduced in § 2. The geometrical arguments, and results, may be translated back into physical language at any stage, but a simple geometrical idea may correspond to a complicated physical one.

Considerable interest has been aroused in questions connected with the singularities of supersonic flow fields. These fall into two classes. The singularities of the limit type are singularities of the transformation from the hodograph plane (or the characteristic plane) to the physical plane, i.e. singularities of the Mach line pattern in the physical plane; in many cases they imply singularities of the velocity gradient. The other class are the singularities of the inverse transformation, and these are called branch-type singularities. Both classes have been investigated by Craggs (1948) under the assumption that the velocity components have third-order derivatives with respect to the space co-ordinates, and vice versa. The arguments employed in the present investigation do not involve such assumptions, and they are therefore used to study the properties of non-analytic solutions occurring in problems of the hyperbolic type.

Except in § 6, it is assumed that the velocity components themselves are continuous. But the velocity gradient may be discontinuous, and such a discontinuity is called a first-order disturbance (see Meyer 1948*a*). Again, the velocity gradient may be continuous, but derivatives of some higher order of the velocity components may be discontinuous, in which case we speak of a higher order disturbance. The variation in strength of disturbances along characteristics is studied in § 4, and the reflexion of first-order disturbances at the sonic line is discussed in § 7.

It is shown in §§ 5 to 5·2 that Craggs's results concerning the singularities of supersonic flows remain valid even when disturbances are present, provided they are not of the first order. When first-order disturbances occur, however, the properties of limit lines are modified considerably (§§ 5·3, 5·4).

In Craggs (1948) the singularities are investigated by the help of the Jacobian of the hodograph transformation. The arguments employed in the present paper do not involve the Jacobian, and cover cases where both the hodograph transformation and the inverse transformation are singular, including simple waves. Only types of singularities of which examples have been reported are discussed in the present paper. More complex types of singularities may, however, occur; examples could be constructed by the help of suitable boundary conditions.

It is commonly accepted that the occurrence of limit lines is connected with the formation of shockwaves. § 6 contains a brief presentation of what is known so far about this problem.

The results concerning limit lines and the formation of shockwaves apply also to steady supersonic, irrotational, isentropic, axially symmetrical flow, except in the neighbourhood of the axis, since the proofs are based on arguments of differential geometry. Similarly, they apply to steady, supersonic, axially symmetrical or two-dimensional flow with vorticity and with entropy changes, provided the vorticity and the entropy gradient are bounded. Methods similar to those employed here may also be applied to the study of one-dimensional, unsteady, isentropic flow, and similar results have been obtained by P. M. Stocker (unpublished). Again, the results concerning limit lines and the formation of shockwaves apply also to unsteady flow with spherical symmetry (except near the centre of symmetry), as well as to one-dimensional, unsteady flow with entropy changes (provided the rate of change of the entropy is bounded), and to the unsteady flow in ducts the cross-section of which varies slowly enough for the variation of the velocity over any cross-section to be negligible.

2. Characteristic co-ordinates

If μ, θ, q, a_s denote the Mach angle, the angle the velocity direction makes with the x-axis, the velocity magnitude, and the critical sonic speed, respectively, then the characteristic equations for the two-dimensional, steady, irrotational, isentropic, supersonic flow of a perfect gas are (Meyer 1948 *b*):

$$dy/dx = \tan(\theta-\mu); \; \theta+t = \alpha = \text{const. on any 'plus' Mach line,*} \tag{1}$$

$$dy/dx = \tan(\theta+\mu); \; \theta-t = \beta = \text{const. on any 'minus' Mach line,} \tag{2}$$

where
$$t(\mu) = \int_{a_s}^{q} \frac{\cot\mu}{q}\,dq, \tag{3}$$

and q and μ are related by Bernoulli's equation

$$q^2\left(\frac{1}{2}+\frac{\sin^2\mu}{\gamma-1}\right) = \frac{\gamma+1}{\gamma-1}a_s^2 = \text{const.} \tag{4}$$

Let the positive direction of a Mach line be defined as that making an acute angle with the stream direction, and let $h_\alpha(\alpha,\beta)\,d\alpha$ and $h_\beta(\alpha,\beta)\,d\beta$ be the elements of length in the positive directions of the minus and plus Mach lines, respectively, in the x, y-plane (the 'physical' plane). The curvatures of the plus and minus Mach lines may then be written

$$\left.\begin{aligned}\kappa_\beta &= \partial(\theta-\mu)/h_\beta\partial\beta = (1-N)/h_\beta,\\ \kappa_\alpha &= \partial(\theta+\mu)/h_\alpha\partial\alpha = (1-N\end{aligned}\right\} \tag{5}$$

respectively, where
$$N = \tfrac{1}{2}(1-d\mu/dt) = \tfrac{1}{4}(\gamma+1)\sec^2\mu > \tfrac{1}{2}, \tag{6}$$
by (4).

Moreover, let
$$H_\alpha = (h_\alpha\sin 2\mu)^{-1}, \quad H_\beta = (h_\beta\sin 2\mu)^{-1}. \tag{7}$$

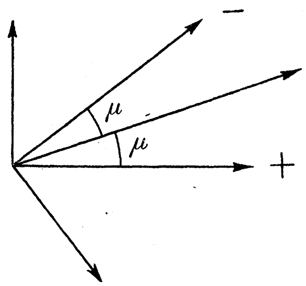

Figure 1

(If a Riemannian geometry is defined in the physical plane by the help of the 'characteristic co-ordinates' α, β, then h_α^2 is a covariant, and H_α^2 the corresponding contravariant, component of the metric tensor. A geometrical interpretation of H_α is as follows. Let the positive direction on the normals to the Mach lines be defined as that making an acute angle with the stream direction (figure 1). Then H_α is the derivative of α in the positive direction of the normal to the plus Mach line, and the element of length in this direction equals $(1/H_\alpha)\,d\alpha$).

Note that the definition of the length parameters h and H, together with the assumption that the velocity components are continuous functions of x and y, implies lemma 1.

* α and β represent the stream direction at the sonic end of the respective Mach lines. The function $P = 1000-t(\mu)$ has been tabulated (Herbert & Older 1946).

Lemma 1. *h_α and H_α are continuous functions, where they are bounded, of β on either side of any plus Mach line, and similarly for h_β, H_β as functions of α and any minus Mach line.*

A physical interpretation of these geometrical parameters of the Mach-line pattern is obtained by noting that the components of the velocity gradient in the positive Mach directions are

$$\left.\begin{aligned} f_\alpha &= \partial q/h_\alpha \partial\alpha = q\sin^2\mu\, H_\alpha, \\ f_\beta &= \partial q/h_\beta \partial\beta = -q\sin^2\mu\, H_\beta, \end{aligned}\right\} \tag{8}$$

respectively, by (3), (1) and (7). The component of the velocity gradient in the stream direction is therefore

$$f_s = \tfrac{1}{2}\sec\mu(f_\alpha + f_\beta) = \tfrac{1}{2}q\tan\mu\sin\mu\,(H_\alpha - H_\beta); \tag{9}$$

the streamline curvature is (by (1), (2) and (7)),

$$\kappa_s = \tfrac{1}{2}\sec\mu(\partial\theta/h_\alpha\partial\alpha + \partial\theta/h_\beta\partial\beta) = \tfrac{1}{2}\sin\mu\,(H_\alpha + H_\beta), \tag{10}$$

and the angle η between the streamline and the isobar at any point is given by

$$\tan\eta = -f_s/(q\kappa_s) = \tan\mu\,(H_\beta - H_\alpha)/(H_\beta + H_\alpha). \tag{11}$$

Note that (from (5)) h_β and h_α are proportional to the radii of curvature of the plus and minus Mach lines, and (from (8)) H_β and H_α are proportional to the components of velocity gradient along the tangents to these lines.

3. The focusing equations

The characteristic equations, (1) and (2), can be replaced by

$$\left.\begin{aligned} \partial x/\partial\beta &= h_\beta\cos(\theta-\mu), \quad & \partial x/\partial\alpha &= h_\alpha\cos(\theta+\mu), \\ \partial y/\partial\beta &= h_\beta\sin(\theta-\mu), \quad & \partial y/\partial\alpha &= h_\alpha\sin(\theta+\mu). \end{aligned}\right\} \tag{12}$$

By equating the mixed second derivatives of x and y, making use of (1), (2) and (6), we obtain the 'focusing equations',*

$$\partial h_\beta/\partial\alpha = N\operatorname{cosec}2\mu\,(h_\beta\cos 2\mu - h_\alpha), \tag{13}$$

$$\partial h_\alpha/\partial\beta = N\operatorname{cosec}2\mu\,(h_\beta - h_\alpha\cos 2\mu). \tag{14}$$

The focusing equations are linear, with coefficients that are known functions of the independent variables. They can be integrated, for example, by Massau's method (Meyer 1948*b*); x and y are then found from (12) (or (15)). A special procedure, however, has to be adopted near a branch line. On the other hand, certain difficulties are avoided which arise with the conventional method, e.g. in the determination of the position of a limit line.

The equations (12) can be inverted to give

$$\left.\begin{aligned} \partial\alpha/\partial x &= -H_\alpha\sin(\theta-\mu), \quad & \partial\alpha/\partial y &= H_\alpha\cos(\theta-\mu), \\ \partial\beta/\partial x &= H_\beta\sin(\theta+\mu), \quad & \partial\beta/\partial y &= -H_\beta\cos(\theta+\mu). \end{aligned}\right\} \tag{15}$$

* A set of equations equivalent to (13) and (14) was established by Nikolsky & Taganoff (1946), without notice of their wider implications. The equations were formally integrated, and so the function in equation (26) below appeared in their paper.

By equating the mixed, second derivatives of α and β, we find that

$$\frac{\partial H_\alpha}{\partial x}\cos(\theta-\mu)+\frac{\partial H_\alpha}{\partial y}\sin(\theta-\mu)=\frac{\partial H_\alpha}{h_\beta\partial\beta}=-H_\alpha(NH_\alpha+(N-1)\,H_\beta\cos 2\mu), \tag{16}$$

$$\frac{\partial H_\beta}{h_\alpha\partial\alpha}=H_\beta(NH_\beta+(N-1)\,H_\alpha\cos 2\mu). \tag{17}$$

3·1. The two systems of equations (13), (14) and (16), (17) are equivalent when h_α, h_β, H_α and H_β are all bounded. The equations (12) hold when h_α and h_β are bounded, and the equations (15) when H_α and H_β are bounded. Hence, provided h_α and H_β are non-zero and bounded,

$$\partial h_\alpha/h_\beta\partial\beta=N(\operatorname{cosec}2\mu-h_\alpha H_\beta\cos 2\mu), \tag{18}$$

$$\partial H_\beta/\partial\alpha=H_\beta(NH_\beta h_\alpha+(N-1)\cot 2\mu), \tag{19}$$

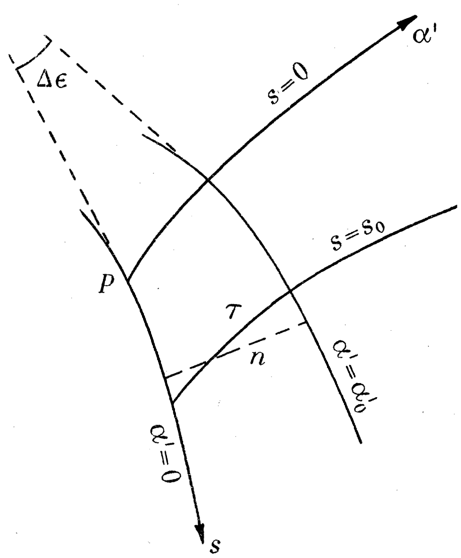

FIGURE 2

for these equations can be deduced (by (7) and (6)) from (13) and (14), as well as from (16) and (17). To prove that (18) holds even where $h_\alpha=H_\beta=0$, we need to show that

$$\partial h_\alpha/h_\beta\partial\beta=N/\sin 2\mu,$$

where $H_\beta=0$, by an argument that is not affected if h_α vanishes as well; a similar argument may serve in the case of equation (19).

Let P be a point where $H_\beta=0$, and take as second characteristic parameter the length s measured from P along the plus Mach line through P, with s increasing in the positive direction (figure 2). Let $\alpha'=\alpha-\alpha(P)$. If we denote by $\tau(s_0)$ the distance, measured along the minus Mach line on which $s=s_0$, between the points where $\alpha'=0$ and $\alpha'=\alpha'_0$, then at P,

$$\partial h_\alpha/h_\beta\partial\beta=\lim_{s_0=0}\lim_{\alpha'_0=0}\frac{\tau(s_0)-\tau(0)}{s_0\alpha'_0}.$$

Now the angle of inclination of a minus Mach line is $(\theta+\mu)$, and $\partial(\theta+\mu)/\partial\alpha'=(1-N)$, which is bounded.* To the first order, $\tau(s_0)$ may therefore be replaced by the length of the

* We assume strictly supersonic conditions ($\mu<\frac{1}{2}\pi$) throughout §§ 3 to 6.

chord of the minus Mach line or, alternatively, by cosec 2μ times the normal distance, $n(s_0)$, between the two plus Mach lines. Moreover, at P, since $H_\beta = 0$, $d\mu/ds = 0$ by (1), (2) and (7), and

$$\partial h_\alpha / h_\beta \partial\beta = \lim_{s_0=0} \lim_{\alpha_0'=0} \frac{n(s_0)-n(0)}{s_0 \alpha_0' \sin 2\mu}.$$

The curvature of the plus Mach line vanishes at P, by (6) and (7), and by Lemma 1 it is a continuous function of α' for fixed s_0. Hence

$$n(s_0) - n(0) = s_0 \tan \Delta\epsilon + \ldots = s_0 \Delta\epsilon + \ldots,$$

where $\Delta\epsilon$ is the angle between the tangents of the plus Mach lines $\alpha' = 0$ and $\alpha' = \alpha_0'$ at their respective points of intersection with the minus Mach line $s = 0$. By (6), $\Delta\epsilon = N\alpha_0' + \ldots$, and hence

$$\partial h\alpha / h_\beta \partial\beta = N/\sin 2\mu$$

at P. (It may be noted that our proof applies also to the case where H_β vanishes identically in a region of the physical plane—so that all plus Mach lines are straight in that region—i.e. to the case of a simple wave.)

Similarly,

$$\partial h_\beta / h_\alpha \partial\alpha = N(h_\beta H_\alpha \cos 2\mu - \operatorname{cosec} 2\mu), \tag{20}$$

$$\partial H_\alpha / \partial\beta = -H_\alpha (N H_\alpha h_\beta + (N-1) \cot 2\mu), \tag{21}$$

when h_β and H_α are bounded.

4. Disturbances

First-order disturbances (see Meyer 1948*a*) are finite* discontinuities of the first-order derivatives (in the physical plane) of q and θ, i.e. discontinuities of H_α or H_β or both, which, according to the theory of characteristics (Courant & Hilbert 1937), persist along the respective Mach lines. It is convenient to associate a disturbance with the family of Mach lines along one of which it persists. Thus

$$\Delta h_\beta = h_{\beta 2} - h_{\beta 1}$$

(suffixes 1, 2 refer to upstream and downstream values, respectively) denotes the strength of a 'minus' disturbance, by lemma 1. By equation (13) and lemma 1,

$$\partial \Delta h_\beta / \partial\alpha = N \cot 2\mu \, \Delta h_\beta.\dagger \tag{22}$$

From α_0 to α the strength of a minus disturbance therefore increases by a growth factor

$$F(\alpha_0, \alpha) = \exp\left[\int_{\alpha_0}^{\alpha} N \cot 2\mu \, d\alpha'\right], \tag{23}$$

where $\beta = \text{const.}$ during the integration, so that $d\mu = \frac{1}{2}(1-2N)\,d\alpha'$ by (1), (2) and (6). Similarly,

$$\partial \Delta h_\alpha / \partial\beta = -N \cot 2\mu \, \Delta h_\alpha, \tag{24}$$

which leads to a growth factor

$$G(\beta_0, \beta) = \exp\left[-\int_{\beta_0}^{\beta} N \cot 2\mu \, d\beta'\right] \tag{25}$$

* Except at isolated points.

† In a simple wave where $H_\alpha = 0$ we must have recourse to equation (20).

along a plus Mach line, from β_0 to β, where now $d\mu = -\frac{1}{2}(1-2N)\,d\beta'$. *Qua* functions of μ both growth factors are therefore identical; they are given by the quotient $f(\mu)/f(\mu_0)$, where

$$f(\mu) = \{[(\gamma - \cos 2\mu)^{\gamma} \sin^{-(\gamma+1)}\mu]^{1/(\gamma-1)} \sec \mu\}^{\frac{1}{2}} \tag{26}$$

($f(\mu)$ has a minimum for $\mu = \frac{1}{4}\pi$, i.e. $M = \sqrt{2}$; see figure 3).

If a disturbance of the nth order is defined as a finite discontinuity of $\partial^{n-1}h_\alpha/\partial\alpha^{n-1}$ or $\partial^{n-1}h_\beta/\partial\beta^{n-1}$, first order, linear, ordinary differential equations are also found as the focusing equations for disturbances of any order, but, in contrast to (22) and (24), they are not homogeneous. A second-order disturbance $\Delta(\partial h_\beta/\partial\beta)$, for example, satisfies the equation

$$\frac{\partial}{\partial\alpha}\Delta\frac{\partial h_\beta}{\partial\beta} = N\cot 2\mu\,\Delta\frac{\partial h_\beta}{\partial\beta} - \left[N^2\operatorname{cosec}^2\mu - \frac{\partial}{\partial\beta}(N\cot 2\mu)\right]\Delta h_\beta$$

(by (13) and (14)), and this integrates to

$$\Delta\frac{\partial h_\beta}{\partial\beta} = CF(\alpha_0, \alpha) - \Delta h_\beta\left[N\cot 2\mu + \int_{\alpha_0}^{\alpha} N^2\operatorname{cosec}^2 2\mu\,d\alpha'\right],$$

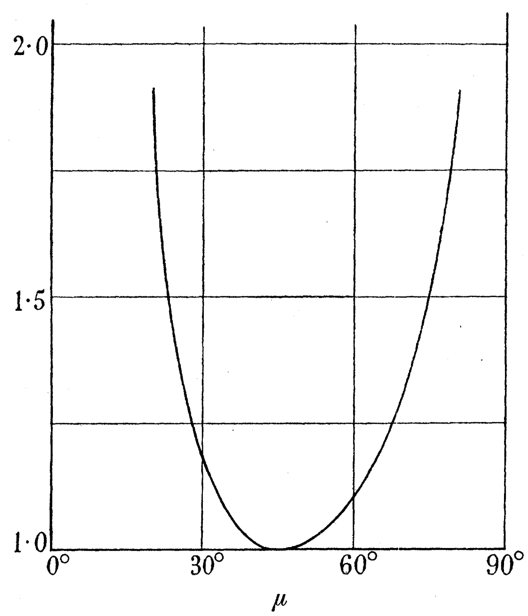

FIGURE 3. The growth factor $f(\mu)/f(\pi/4)$, for $\gamma = 7/5$.

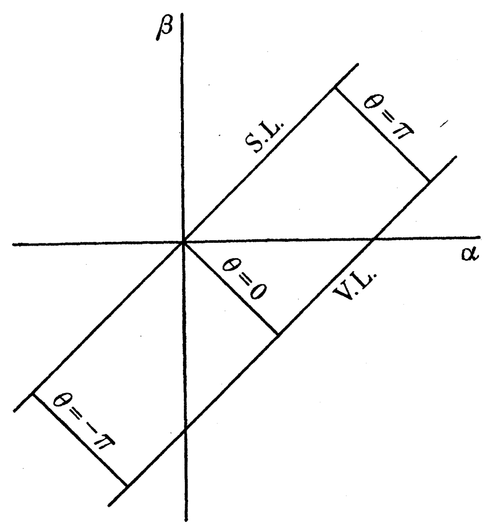

FIGURE 4. Characteristic plane. S.L. sonic line ($\mu = \frac{1}{2}\pi$); V.L. vacuum line ($\mu = 0$).

since $\partial\mu/\partial\beta = -\partial\mu/\partial\alpha$, by equations (1) to (4); C is a constant of integration. The focusing equation for a disturbance of order n (>1) is distinguished from an equation of the form (22) or (24) by a set of terms linear in the disturbances of order $<n$. A disturbance of any order therefore implies disturbances of all higher orders. On the other hand, if no disturbances of order $<n$ are present on a Mach line, a disturbance of order n satisfies an equation of the same form as a first-order disturbance, and has the same growth factor.

In the investigation of the supersonic flow of inviscid fluids, disturbances are encountered in two ways: first, in connexion with simple waves and with boundary conditions that exclude analytic solutions, and secondly, in connexion with virtual perturbations ('wavelet' arguments) or discontinuous approximations to analytic solutions (finite difference methods). In both cases it is useful to distinguish clearly between the properties of disturbances of particular orders—which may be introduced as accidental byproducts of the neglect of boundary layers, or of particular methods of mathematical treatment—and the properties common to disturbances of all orders, which must be regarded as indicative of properties of supersonic flows as such. The growth factor (26) belongs to the latter type; it describes the steepening of any wave front* however weak the disturbance may be that marks it.

5. Singularities

The role of first-order disturbances in connexion with limit lines and branch lines differs from that of disturbances of any higher order. We shall begin by deducing certain properties of limit lines and branch lines under the assumption that h_α, h_β, H_α and H_β are continuous (where they are bounded). In §§ 5·3 and 5·4 we shall discuss how those results are affected when first-order disturbances are present.

The singularities of the transformation from the physical plane to the hodograph plane, and vice versa, have been investigated in Craggs (1948). The co-ordinate net in the α, β-plane ('characteristic' plane; figure 4) corresponds to the well-known epicycloid net in the hodograph plane. It is obvious that the transformation of the supersonic region of the hodograph plane into the characteristic plane is 1–1 except at the sonic line and at the vacuum line. Most properties of the transformation from the physical plane to the characteristic plane are therefore analogous to those of the hodograph transformation, and can be obtained from knowledge of the Jacobian of the latter transformation. In simple waves, however, lines are encountered (cf. § 5·4) that possess all the properties of limit lines except the one by which limit lines are defined in Craggs (1948), viz. the property that the Jacobian of the transformation vanishes. Even in these cases, where the Jacobian fails to provide any information on the properties of the transformation, the length parameters introduced in § 2 describe them fully (§ 5·4). Moreover, limit lines as well as branch lines may be divided naturally into two classes corresponding to the Mach-line families. We shall therefore define them directly in terms of the length parameters.

5·1. We define a branch point of the first order as a point where one, and only one, of the length parameters H_α, H_β vanishes. (Such a point is a branch point in the sense of Craggs (1948), for the Jacobian of the transformation from the physical plane to the characteristic plane is

$$J = H_\alpha H_\beta \sin 2\mu.) \tag{27}$$

Theorem 1. *Any branch point of the first order lies on a branch line,*† *which cannot end inside a supersonic region.*

Let $H_\alpha = 0$ at a point P in the physical plane. If it were not true that H_α vanishes identically on the plus Mach line through P there would be a point Q on this line such that

$$H_\alpha(Q) \neq 0, \tag{28}$$

* The term 'wave front' is used, in the first place, for the frontier of a region occupied by a simple wave, and is then generalized to denote the frontier of a region in which the velocity components are analytic in the space co-ordinates; in any case the term is used only when the frontier is not a shock wave (cf. Courant & Hilbert 1937). By the 'steepening' of a wave front, an increase is meant in the magnitude of the lowest order discontinuity that occurs; and the number, n, of this order may be regarded as a measure of the 'weakness' of the disturbance.

† The existence of branch lines was discovered by Lighthill (1947); the first part of the theorem is due to Craggs (1948).

but $$H_\alpha(P) = 0. \tag{29}$$

Now, equation (14) (as well as (18)) integrates to

$$h_\alpha(P) = G(\beta_Q, \beta_P)\left[h_\alpha(Q) + \int_Q^P \frac{N \operatorname{cosec} 2\mu}{G(\beta_Q, \beta)} h_\beta \, d\beta\right], \tag{30}$$

where the integral is to be evaluated along the plus Mach line. (Use is made of (25).) $G(\beta_Q, \beta)$ is non-zero, and $N \operatorname{cosec} 2\mu$ is bounded, for $0 < \mu < \frac{1}{2}\pi$, and hence (28) and (29) are incompatible if the Mach-line segment PQ is of finite length.

A line where $H_\alpha = 0$ is therefore a plus Mach line, and a line where $H_\beta = 0$ is a minus Mach line; we shall call the former a plus branch line and the latter a minus branch line. Various properties of branch lines are discussed in Craggs (1948). An important property of all branch lines encountered to date is that $\partial H_\alpha / h_\alpha \partial\alpha$ is bounded on all plus branch lines, and $\partial H_\beta / h_\beta \partial\beta$ on all minus branch lines; it follows that H_α varies like $(\alpha - \alpha_0)^{\frac{1}{2}}$ on any minus Mach line near a plus branch line on which $\alpha = \alpha_0$ and $\partial H_\alpha / h_\alpha \partial\alpha \neq 0$.

If $H_\alpha = \partial H_\alpha / h_\alpha \partial\alpha = 0$ we may speak of a double branch line (its properties are discussed in Craggs (1948)), and if further derivatives vanish, of a multiple branch line. A simple wave is a region covered by branch lines of one family.

A point where $H_\alpha = H_\beta = 0$ may be termed a branch point of the second order; examples are the sonic point of a straight streamline (Lighthill 1947) and the point where a branch line meets a free streamline (by (9)),* e.g. a jet boundary. A region of uniform flow is covered by branch points of the second order.

The role of branch lines in Massau's method of step-by-step integration of the equations of motion is discussed in the appendix.

5·2. We define a limit point of the first order as a point where one, and only one, of the length parameters h_α, h_β vanishes. (Such a point is a limit point in the sense of Craggs (1948), for the Jacobian of the transformation from the characteristic plane to the physical plane is

$$J' = h_\alpha h_\beta \sin 2\mu = J^{-1}.) \tag{31}$$

Theorem 2. *Any limit point of the first order lies on a limit line,† which cannot end inside a supersonic region except at a limit point of higher order.*

This follows from (18) (or (20)) and the implicit function theorem. Note that the assumption is implied that h_α (or h_β) is continuous. No example is known of a higher order limit point.‡

A limit line where $h_\alpha = 0$ ($h_\beta = 0$) will be termed a plus (minus) limit line. On a plus limit line the velocity gradient is perpendicular to the plus Mach lines, by (8), and we conclude from Craggs (1948) that it envelops plus Mach lines. (This, and the other properties of limit lines derived in Craggs (1948), can be proved also when the Jacobian does not exist (cf. § 5·4).)

* The author is indebted to Mr Lighthill for pointing out this example.

† The first part of the theorem is due to Craggs (1948). A cusp of a limit line is not interpreted in theorem 2 as a point where it ends.

‡ [*Note added in proof.*] Examples have been found in one-dimensional unsteady flow by Burgers (1948) and P. M. Stocker (unpublished). In both cases the singularity is of one of the types discussed in Craggs (1948); two singular lines intersect in the characteristic plane, both limit lines are cusped, and a sector of the physical plane is covered four times.

A point where $h_\alpha = \partial h_\alpha/\partial\alpha = 0$, $\partial^2 h_\alpha/\partial\alpha^2 \neq 0$, is a cusp of a plus limit line (cf. Craggs (1948)). More generally, let the singular line $h_\alpha = 0$ in the characteristic plane be given by the equation $\beta = \beta_l(\alpha)$, and the limit line in the physical plane by the equations $x = x_l(\alpha)$ and $y = y_l(\alpha)$. Then any point where the singular line touches a minus Mach line (so that $d\beta_l/d\alpha = 0$) corresponds to a singular point of the limit line ($dx_l/d\alpha = dy_l/d\alpha = 0$ by (12)); when $d\beta_l/d\alpha$ changes sign this point is a cusp. When $h_\alpha = 0$ on a finite segment of a minus Mach line in the characteristic plane, then the corresponding segment of the limit line is reduced to a point (since its length is $\int h_\alpha\, d\alpha$), and at that point θ and t change discontinuously (by (1) and (2)); the best-known example is the centre of a simple wave (corner of a Prandtl-Meyer expansion).*

THEOREM 3. *If a Mach line has two limit points of its family they must be separated on it*† *by an odd number of points where the Mach line is cusped.*‡

Proof. By lemma 1 and theorem 1, h_α can change sign on a plus Mach line only where $h_\alpha = 0$. At any such point $\partial h_\alpha/h_\beta\,\partial\beta > 0$, by (18), if $h_\beta \neq 0$. Hence, if we follow a plus Mach line from a point where $h_\alpha = 0$, either exclusively in the positive direction, or exclusively in the negative direction, we cannot arrive at another point where $h_\alpha = 0$.

Near a cusp of a Mach line, however, the positive direction on it, as defined in § 2, points either towards the cusp on both branches, or away from it on both branches. If it is a plus Mach line with the positive direction pointing towards the cusp, say, and with $h_\alpha > 0$ at the cusp, then equation (18) does not exclude the occurrence of a zero of h_α on each of the two branches.

On the other hand, if a plus Mach line has just two cusps and if h_α has the same sign at both of them, then h_α can vanish only once on the Mach line (by (18)). The occurrence of two points where $h_\alpha = 0$ on the same plus Mach line therefore requires an odd number of cusps of the Mach line between them.

The cusp of a Mach line is a limit point of the other family, by (12). If a Mach line is not cusped at a limit point of the other family, then the limit line is cusped (cf. above) (such a point might be counted as an even number of limit points in the physical plane, in view of the two-point contact, or even number point-contact, between the Mach line and the singular line in the characteristic plane).

THEOREM 4. *If a supersonic region is extended far enough, any Mach line that is not a branch line must meet a limit line.*

For simple waves this is a well-known result (Shepherdson 1946, 1947; Courant & Friedrichs 1948). That it is true in general follows from the fact that any Mach line that is not cusped, and not a branch line, must have a limit point of its family if the Mach line extends over a sufficient distance, by virtue of (30) and of the corresponding equation for h_β, since $F(\alpha_0, \alpha)$, $G(\beta_0, \beta)$, N, and $\sin 2\mu$ are all positive. It may, of course, be possible to consider only a portion of the supersonic field (supposed bounded, at any rate in part,

* In that case h_α is discontinuous, since a simple wave is flanked by regions of uniform flow; the centre is an isolated limit point of the first order (cf. § 5·3).

† Provided no point of the Mach line is a limit point of higher order.

‡ The author is indebted to Mr Stocker for an important contribution to this result.

by solid boundaries) in which limit lines do not occur; limit lines always occur, however, in the part of the supersonic field that is the extension of the portion considered.

The theorems 1 to 4 provide useful criteria for the prediction of limit lines. For instance, if we follow a plus Mach line in the negative direction to the boundary of the region of flow, and if $h_\alpha > 0$ there, then (by (18)) the first limit point we can possibly meet on retracing this Mach line back into the region of flow must be a zero of h_β. In other cases it may be easier to argue in a different way; we may be able to satisfy ourselves that a Mach line must have a limit point of its family beyond the given boundary streamline, if the field of flow is extended in that direction. If the boundary streamline is of bounded curvature it cannot meet the limit line,* which is therefore confined to one side of the boundary streamline as long as the latter does not meet a branch line (or a first order disturbance, cf. § 5·3) of the same family. A limit line cannot meet a branch line of the same family, but may approach it asymptotically, and may approach the two sides of a branch line on different sides of the boundary streamline. Examples are easily constructed of simple waves that exhibit these properties (Shepherdson 1946, 1947).

5·3. We now proceed to investigate how the above theorems are modified if first-order disturbances are present. By lemma 1, theorem 1 still holds on either side of any Mach line. If H_α is discontinuous at a point P in such a way that it tends to zero as P is approached in some direction, then H_α must vanish identically on that side of the plus Mach line through P on which it vanishes at P, and it cannot vanish on the other side of this Mach line. Such a line may be called a 'half' branch line. It occurs frequently in connexion with simple waves; for example, in the case of a jet (figure 5), with uniform flow upstream of the mouth, the Mach line separating the regions S_{-1}, G_1, S_{-2} from the regions U_1, S_{+1}, U_3 and U_2, S_{+2}, U_4, respectively, are plus half-branch lines. In a nozzle corrected to give a uniform flow (figure 6), the regions S_+, S_- are simple waves; their upstream borders are half-branch lines if the velocity gradient on the axis is discontinuous at the point A (by (9) and (10)); when approached from the upstream side they present themselves as single branch lines if this gradient is continuous, and multiple branch lines if the higher derivatives of the velocity components, up to some order, are continuous as well. These distinctions are of importance for the accurate design of wind-tunnel nozzles (cf. appendix).

A limit line must be cut off, or at least interrupted, by a first-order disturbance of its own family. For it is shown in § 4 that h_α must be discontinuous at every point of a plus Mach line, if it is discontinuous at one point. Theorems 3 and 4, however, remain valid on each side of any Mach line. Examples of Mach lines, the two sides of which meet a limit line of the same family at different points (so that the limit line is not a continuous curve), are found in Shepherdson (1946, 1947); in such cases the limit line starting on the downstream side of the Mach line may be interpreted as the continuation of the limit line ending on the upstream side.

There are also examples of a limit line starting at some point of a Mach line carrying a disturbance, with no upstream branch of the same limit line present. In Pack (1948), for instance, where two cases of jets of the type indicated in figure 5 have been treated by numerical methods, limit lines are found starting on the border between the regions S_{+3} and

* By (10); the streamline through the cusp of a limit line has infinite curvature (see also Craggs 1948).

G_4, and S_{+2} and G_2, respectively. These limit lines start on the borders of simple-wave regions, but do not themselves form part of such regions. They are examples of limit lines starting at a point of a half-branch line; in such cases h_α is both zero and infinite at the same point, being zero on one side of the plus Mach line through the point and infinite on the other side. Limit lines that start in the middle of the flow do not occur only in connexion with simple-wave regions; examples have been found in axially symmetrical flow (Meyer, 1948*a*; Johannesen & Meyer 1949).

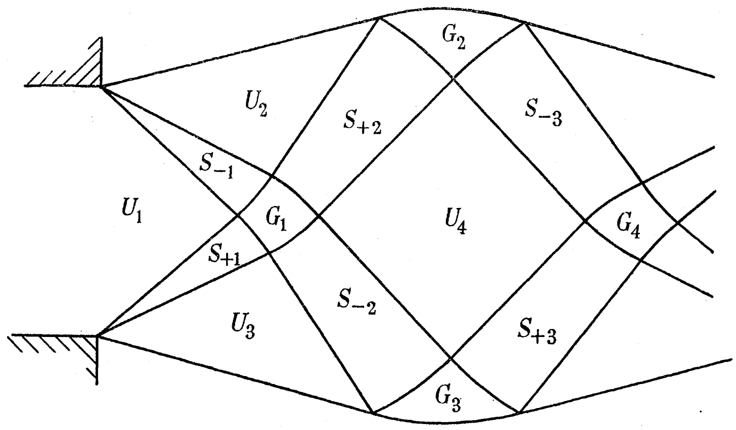

FIGURE 5

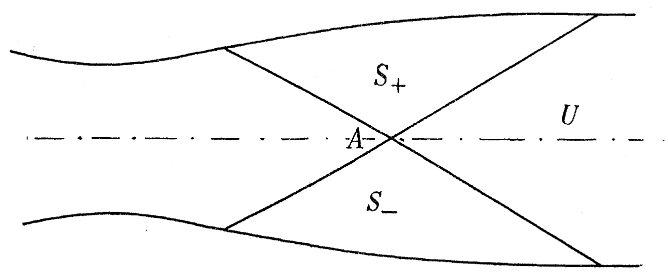

FIGURE 6

5·4. We have seen that the singularities of the transformation from the physical plane to the characteristic plane resulting from the simultaneous occurrence of limit lines, branch lines, and first-order disturbances, appear more complex than the singularities of the first order treated in Craggs (1948); some of them cannot be discussed in terms of the Jacobian. In order to show how the geometrical features of such singularities can be deduced from the behaviour of the length parameters, let us consider the example of a simple wave, covered by plus branch lines and containing a minus limit line starting at a point where h_β is discontinuous.

Since H_α vanishes identically in the region, the minus Mach lines are straight (by (5) and (7)), and they are at the same time the isobars and isoclines. The curvatures of the plus Mach lines and the streamlines, respectively (MM' and SS', in figure 7), are infinite at the limit line (AB), by (5) and (10). Both lines are cusped, for on the streamlines, for example,

$$h_\alpha d\alpha = h_\beta d\beta \tag{32}$$

(since the streamlines bisect the angle between the positive directions on the respective Mach lines), and hence, by (12),

$$\left.\begin{aligned} dx/d\beta &= \cos(\theta+\mu)\, h_\alpha d\alpha/d\beta + h_\beta \cos(\theta-\mu) = h_\beta[\cos(\theta+\mu)+\cos(\theta-\mu)],\\ dy/d\beta &= h_\beta[\sin(\theta+\mu)+\sin(\theta-\mu)]. \end{aligned}\right\} \tag{33}$$

It follows that both $dx/d\beta$ and $dy/d\beta$ vanish at the limit line, and that at least one of them changes sign (except where the limit line is itself cusped). It can be proved similarly that the limit line is an envelope of the minus Mach lines. By (8), the velocity gradient is infinite and perpendicular to the minus Mach lines.

On one side (we shall call this side II) of the minus Mach line DAC the parameter h_β vanishes at A; for definiteness let $h_\beta > 0$ on DA, and $h_\beta < 0$ on AC. Since h_β is discontinuous across DAC, $h_\beta \neq 0$ on the other side (side I) at A, and, by lemma 1, we may take C and D sufficiently close to A so that $h_\beta \neq 0$ between C and D on side I. Let us assume that $h_\beta > 0$ on

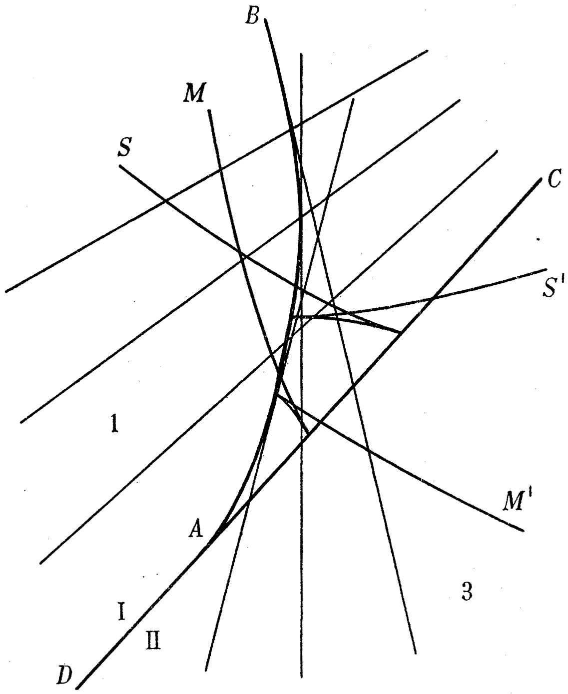

FIGURE 7

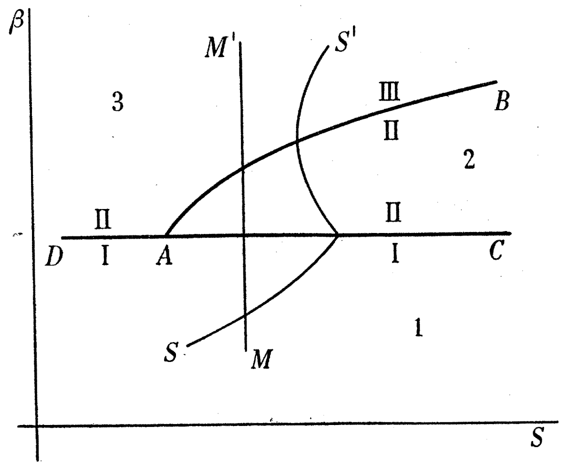

FIGURE 8

side I of the Mach line DAC; it will appear immediately that h_β must, in fact, in the case shown in figure 7, have the same sign on side I as on side II of the segment DA. It follows from (5), (10) and (8) that the curvatures of the plus Mach lines and streamlines, respectively, and the component in the plus Mach direction of the velocity gradient are discontinuous across the Mach-line segment DA; they are finite on both sides, and none of them changes sign. 'Across'* the segment AC they are all discontinuous; they are finite on both sides, and change sign. On the streamlines, $dx/d\beta$ and $dy/d\beta$ also change sign 'across' AC, by (33), but θ is continuous; the streamlines show the same behaviour as at a limit line, except that the curvature is finite. The same is true for the plus Mach lines. The Mach-line segment AC is therefore an edge of regression of the transformation, but not a limit line in the strict sense. The streamline (and, similarly, the plus Mach line) through A has finite curvature on side I, and infinite curvature on the other side; it is not cusped.

The geometrical properties in the large of the transformation are the same as those of a field of flow containing a limit line with a cusp (Craggs 1948). The Mach-line pattern

* In figure 7, the region between AB and AC is triply covered, so the sides I and II of AC coincide in the figure as drawn on an *unfolded* sheet of paper.

covers a part of the physical plane with three sheets. On sheet 1 h_β is positive. It changes sign discontinuously, from side I to side II of the edge of regression, and on sheet 2, between the edge of regression and the limit line, $h_\beta < 0$. From side II to side III of the limit line h_β changes sign continuously, passing through zero, and on sheet 3, h_β is again positive. H_α vanishes throughout; the Jacobian (27) vanishes everywhere except at the limit line, where it is indeterminate. The 'folding' of the Mach-line pattern, and the appearance of multiply-covered regions, is therefore connected primarily with the change of sign of a length parameter h rather than with that of the Jacobian of the hodograph transformation.

A (non-singular) characteristic plane can be constructed in which β and the length s on the minus Mach line DAC (in place of α), measured in the positive direction, are rectangular Cartesian co-ordinates. Conditions in this plane are indicated in figure 8.

The occurrence, in the physical plane, of edges of regression that are not limit lines need not be connected with a simple wave. When a limit line starts at a point where a length parameter h is discontinuous, then a part of a Mach line through that point must be an edge of regression. For at such a point h must change sign on one side, and only on one side, of one of the Mach lines. In such cases the streamline curvature need not change sign at the edge of regression. The isobars are no longer coincident with Mach lines, as in a simple wave, and their slope changes discontinuously, by (11), at every Mach line carrying a first-order disturbance (the same is true for the isoclines). Conditions in the physical plane and in the α, β-plane are similar to those indicated in figures 7 and 8.

It is evident that a point of regression of a Mach line must be counted as a cusp in the sense of theorem 3.

6. The start of an oblique shockwave

The conjecture has been put forward by Riemann (1860), and adopted generally since, that the occurrence of a limit line leads to the formation of a shockwave. Tollmien (1947) has attempted to prove that an oblique shockwave, if it starts in the middle of the flow with zero initial strength, must start at the cusp of a limit line. His argument can be simplified and extended to apply to singularities of the type described in § 5·4, as follows.

Let suffixes 1, 2 denote values immediately upstream and downstream of the shockwave, respectively. Let $(\omega+\theta_1)$ be the angle of inclination of the shockwave, and $\delta = \theta_2-\theta_1$ the angle through which the stream is deflected. For definiteness, assume that $\delta \geqslant 0$; then $0<\omega<\frac{1}{2}\pi$, the shockfront touches a minus Mach line where its strength vanishes, and for a weak shockwave (Lighthill 1944)

$$\mu_2-\mu_1 = \tfrac{1}{2}\sec^2\mu_1(2\sin^2\mu_1+\gamma-1)\,\delta+O(\delta^2), \tag{34}$$

$$\omega-\mu_1 = N_1\delta+O(\delta^2). \tag{35}$$

It is well known that the equations for the change in q and θ across a weak shockwave differ from those for the change through a simple wave by terms of the order of δ^3. Either $(\alpha_2-\alpha_1)$ or $(\beta_2-\beta_1)$ must therefore be of that order. It follows from (1), (6) and (34) that $\alpha_2-\alpha_1 = O(\delta^2)$ at most. From (1) and (2), $\alpha+\beta = 2\theta$, so the discontinuity in $\alpha+\beta$ is 2δ. Hence

$$\alpha_2-\alpha_1 = O(\delta^3), \quad \beta_2-\beta_1 = 2\delta+O(\delta^3). \tag{36}$$

Let τ denote the arc length on the shockwave, measured so that it increases in the direction making the smaller angle with the stream direction. Then, at any point of the shockwave, and on both sides of it,

$$dx/d\tau = \cos(\theta+\mu)\, h_\alpha\, d\alpha/d\tau + \cos(\theta-\mu)\, h_\beta\, d\beta/d\tau = \cos(\omega+\theta_1),$$

$$dy/d\tau = \sin(\theta+\mu)\, h_\alpha\, d\alpha/d\tau + \sin(\theta-\mu)\, h_\beta\, d\beta/d\tau = \sin(\omega+\theta_1),$$

by (12). Hence, on the upstream side,

$$h_{\beta 1}\, d\beta_1/d\tau = \sin(\mu_1-\omega)/\sin 2\mu_1, \quad h_{\alpha 1}\, d\alpha_1/d\tau = \sin(\mu_1+\omega)/\sin 2\mu_1, \tag{37}$$

and on the downstream side,

$$h_{\beta 2}\, d\beta_2/d\tau = \sin(\mu_2-\omega+\delta)/\sin 2\mu_2, \quad h_{\alpha 2}\, d\alpha_2/d\tau = \sin(\mu_2+\omega-\delta)/\sin 2\mu_2. \tag{38}$$

If the shockwave starts with zero strength, $\delta = 0$, and hence

$$h_\alpha\, d\alpha/d\tau = 1, \quad h_\beta\, d\beta/d\tau = 0, \tag{39}$$

at the point where it starts, on both sides. It follows by (36) that either $h_{\alpha 1} = h_{\alpha 2}$, or $H_{\alpha 1} = H_{\alpha 2} = 0$ at this point. If we assume that $d\delta/d\tau$ does not vanish there, it follows also that either $h_{\beta 2} = 0$, or $h_{\beta 1} = 0$, or both.

By differentiating the equations (37) and (38) with respect to τ, we may extend the argument in either of two ways. We may determine the initial value of $d\delta/d\tau$, for any one of the possible types of singularity in terms of the values of the length parameters and of their derivatives. We may also calculate the initial curvature of the shockwave,

$$d(\omega+\theta_1)/d\tau = \kappa_{\alpha 1} + \tfrac{1}{2} N_1\, d(\beta_2+\beta_1)/d\tau$$

(use is made of (5), (39), (6) and (35)), and the initial values of higher derivatives of both δ and $(\omega+\theta_1)$. In this way we may try to construct a shockwave solution for any given type of singularity of the Mach-line pattern, as has been done in Tollmien (1947) for one case involving a cusp of a limit line. Alternatively, we may study the case where $\delta = d\delta/d\tau = 0$, $d^2\delta/d\tau^2 \neq 0$, at the point where the shockwave starts, and thus seek to prove, eventually, that any shockwave must start at a limit point if it starts with zero strength at a point in a supersonic region and if its strength is an analytic function of τ near the point where it starts.

It should be noticed, however, that our argument is based on the fact that the shockwave equations imply an infinite value of the component of the velocity gradient normal to one of the Mach directions when $\delta \to 0$. The shockwave equations, in turn, are based on the assumption that the width of the shockwave is negligible compared with the macroscopic dimensions of the problem; this is true for shockwaves of finite strength, but not for very weak ones (Taylor 1910). No case is reported where the position of the shockwave has been determined, for the same problem, both from experiment and from the theory of inviscid fluid motion and the shockwave equations.

7. The reflexion of first-order disturbances at the sonic line

To complete our investigation of first-order disturbances, we discuss their reflexion at the sonic line. The case arises in a nozzle with a sharp-edged throat, if the sonic velocity is reached on the wall at a small distance upstream of the throat (figure 9). Another example is the flow through a sharp-edged slit from one container to another (figure 10), if the ratio of the pressures in the respective containers is big enough to generate a supersonic jet, but too small to deflect the boundary streamline, at the edge, through half its angle of inclination just upstream of the edge.*

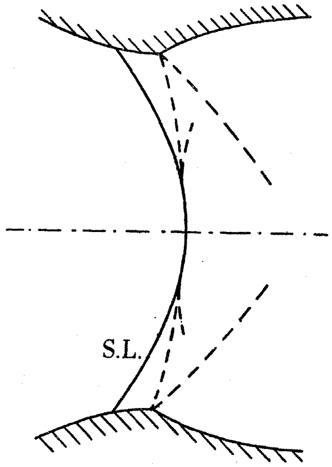

FIGURE 9

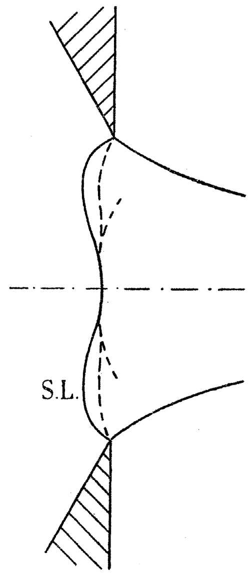

FIGURE 10

The most familiar types of reflexion at boundaries of supersonic regions are that at a free streamline, e.g. a jet boundary, where $f_s = 0$ and hence, by (9), $\Delta H_\alpha = \Delta H_\beta$; and that at a fixed streamline with continuous curvature, e.g. a body contour, where $\Delta H_\alpha = -\Delta H_\beta$, by (10).

THEOREM 5. *If the velocity gradient and the radius of curvature of the streamline are both bounded at the sonic line, then the reflexion of a first-order disturbance at the sonic line is of the same basic type as that at a free streamline. The slope of the sonic line is continuous. Interpreted as a discontinuity of h, the disturbance reaches the same value at the points of intersection of any one isobar with the two Mach lines through the point of reflexion.*

Proof. By (8), (9) and (10),

$$H_\alpha - H_\beta = O(\cos\mu), \quad H_\alpha = H_\beta = O(1) \tag{40}$$

at the sonic line.† If the disturbance is carried to the sonic line on, say, a plus Mach line,

$$\Delta H_\alpha = H_{\alpha 2} - H_{\alpha 1} = -H_{\alpha 1} H_{\alpha 2} \Delta h_\alpha \sin 2\mu = O(\cos^{\frac{1}{2}}\mu), \tag{41}$$

by (7), (25), (26) and (40) (suffixes 1, 2 refer to upstream and downstream values, respectively); the disturbance does not lead to a discontinuity of H on the sonic line. By (40),

$$\Delta H_\alpha - \Delta H_\beta = O(\cos\mu) \tag{42}$$

at most, and by (7),

$$\Delta h_\alpha - \Delta h_\beta = O(1). \tag{43}$$

* This latter condition ensures that the downstream border of the centred expansion round the corner meets the sonic line.

† So the Jacobian (27), but not (31), vanishes at the sonic line.

It is shown in § 4 that the growth factors of Δh_α and Δh_β are the same function of μ; so therefore are $F(\alpha_0, \alpha)\cos^{\frac{1}{2}}\mu$ and $G(\beta_0, \beta)\cos^{\frac{1}{2}}\mu$. Hence, if Δh_α and Δh_β were not equal, for equal values of μ on their respective Mach lines, (43) could not be true. It follows from (11) and (7) that $\tan\eta$ is continuous on the sonic line.

The reflexion at the sonic line has certain features, however, which distinguish it from that at a free streamline. The discontinuities of f_s and of the isobar slope are $O(\sec^{\frac{1}{2}}\mu)$ on both Mach lines through the point of reflexion; f_s and $\tan\eta$ are continuous on the sonic line itself.

A Schlieren photograph showing the reflexion at the sonic line, as under theorem 5, is found in Prandtl (1936).

The question arises how theorem 5 is affected if the assumption is abandoned that the velocity gradient is bounded at the sonic line.* In particular, can a first-order disturbance lead to a singularity of the velocity gradient there, so that the velocity gradient is bounded on the upstream side, but not on the downstream side, of the Mach line that carries the disturbance? It can be shown that if (40) holds on the upstream side of the Mach line, and if the reflexion is not as under theorem 5, then the velocity gradient must be infinite at the sonic line, on the downstream side, and the slope of the sonic line must change discontinuously so that it becomes perpendicular to the local stream direction.

The author wishes to thank Professor Goldstein for many valuable criticisms.

Appendix note on the numerical integration of the characteristic equations

When the equations (1) and (2) are integrated step by step (Temple 1946), the first of each of the equations (1) and (2) is replaced by a difference equation, and the question arises, what values, exactly, of $\tan(\theta-\mu)$ and $\tan(\theta+\mu)$ are best employed in such equations. Holt (1949) has shown that an error of the order of the step size, δ, is made in every step if the values at either of the end-points of each elementary Mach-line segment are substituted. He has also pointed out that the error made in every step is of the order of δ^2 if, instead, the arithmetic mean is employed of the values at the two end-points of each elementary Mach-line segment. If terms $O(\delta^2)$ are neglected, it is, of course, the same whether $\tan(\theta-\mu)$ is replaced by

$$\bar{\sigma} = \tfrac{1}{2}\{\tan(\theta_1-\mu_1)+\tan(\theta_2-\mu_2)\} \tag{44}$$

(suffixes 1, 2 and m denote values taken at the two end-points, and at the midpoint, respectively, of a plus Mach-line segment of length δ in the characteristic plane), or by

$$\tan[\tfrac{1}{2}(\theta_1+\theta_2-\mu_1-\mu_2)],$$

or by

$$\tan(\theta_m-\mu_m),$$

as suggested in Temple (1946).

It will now be shown that a different procedure is required near a branch line, if the same accuracy is to be achieved. Let point 1 be a minus branch point ($H_\beta = 0$); we assume that $\partial H_\beta/h_\beta\partial\beta$ is bounded (§ 5·1). If $\partial H_\beta/h_\beta\partial\beta \neq 0$ at 1, we may put

$$H_\beta = c(\beta-\beta_1)^{\frac{1}{2}}[1+d(\beta-\beta_1)+O\{(\beta-\beta_1)^2\}]\dagger \tag{45}$$

* The assumption that the streamline curvature is non-zero excludes a branch point (cf. Craggs 1948).

† Such a series is appropriate for the branch line of Lighthill (1947). In general, the form of the expression in the square bracket depends on the boundary conditions, and may be determined from them by the help of (8), (9) and (10). It can be shown, however, that (46) holds (except for the error term, which in any case is $o(\beta_2-\beta_1)$) near any single minus branch line, on which $\partial H_\beta/h_\beta\partial\beta$ is bounded.

near 1, on the plus Mach line 1, 2; c and d are constants. By (12), (7) and (45),

$$y_2-y_1 = \pm\int_{\beta_1}^{\beta_2} \sin(\theta-\mu)\, h_\beta\, d\beta$$
$$= \pm\frac{2\sin(\theta_1-\mu_1)}{c\sin 2\mu_1}(\beta_2-\beta_1)^{\frac{1}{2}}\left[1+\tfrac{1}{3}(\beta_2-\beta_1)\left\{\cot(\theta_1-\mu_1)\frac{\partial(\theta-\mu)}{\partial\beta} - d - 2\cot 2\mu_1\,\partial\mu/\partial\beta\right\} + O\{(\beta_2-\beta_1)^2\}\right],$$

where the derivatives are taken at the point 1, and similarly,

$$x_2-x_1 = \pm\frac{2\cos(\theta_1-\mu_1)}{c\sin 2\mu_1}(\beta_2-\beta_1)^{\frac{1}{2}}\left[1-\tfrac{1}{3}(\beta_2-\beta_1)\left\{\tan(\theta_1-\mu_1)\frac{\partial(\theta-\mu)}{\partial\beta} + d + 2\cot 2\mu_1\,\partial\mu/\partial\beta\right\} + O\{(\beta_2-\beta_1)^2\}\right].$$

The mean slope of the Mach-line segment 12 is therefore

$$\bar{\sigma}_b = \frac{y_2-y_1}{x_2-x_1} = \tan(\theta_1-\mu_1)\left[1+\tfrac{1}{3}(\beta_2-\beta_1)\{\cot(\theta_1-\mu_1)+\tan(\theta_1-\mu_1)\}\frac{\partial(\theta-\mu)}{\partial\beta} + O\{(\beta_2-\beta_1)^2\}\right],$$

which may be written

$$\bar{\sigma}_b = \tfrac{1}{3}[2\tan(\theta_1-\mu_1)+\tan(\theta_2-\mu_2)] + O\{(\beta_2-\beta_1)^2\}. \tag{46}$$

Similarly, if point 1 is a double-minus branch point such that

$$H_\beta = c(\beta-\beta_1)^{\frac{2}{3}}[1+d(\beta-\beta_1)+O\{(\beta-\beta_1)^2\}]$$

near 1, on the plus Mach line, we find that the mean slope of the Mach-line segment 12 is

$$\bar{\sigma}_{2b} = \tfrac{1}{4}[3\tan(\theta_1-\mu_1)+\tan(\theta_2-\mu_2)] + O\{(\beta_2-\beta_1)^2\}.$$

Analogous results are found for the minus Mach-line segments adjacent to a plus branch line.

These corrections may have a noticeable effect on the numerical results even when the step size is small. For if the length of a plus Mach-line segment in the characteristic plane is $\beta_2-\beta_1 = \delta$, the length of the corresponding segment in the physical plane is $\delta/(H_\beta \sin 2\mu)$.

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